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# GREEN'S FUNCTIONS FOR TRANSVERSELY ISOTROPIC PIEZOELECTRIC SOLIDS

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**Abstract**—Closed-form expressions are obtained for the infinite-body Green's functions for a transversely isotropic piezoelectric medium. The four Green's functions represent the coupled elastic and electric response to an applied point force or point charge. The Green's functions are obtained using a formulation where the three displacements and the electric potential are derivable from two potential functions. When piezoelectric coupling is absent, the results reduce to those for uncoupled elasticity and electrostatics. Copyright © 1996 Elsevier Science Ltd

## 1. INTRODUCTION

Green's functions play an important role in the solution of numerous problems in the mechanics and physics of solids. They are the heart of singular integral equation methods such as the boundary element method. Key contributions to the study of elastostatic Green's functions in isotropic and anisotropic media are due to Lord Kelvin (1882), Freedholm (1990), Lifshitz and Rozentsveig (1947), Kröner (1953), Synge (1957), Willis (1965), Mura and Kinoshita (1971), and Pan and Chou (1976). Recently, Wang (1995) has obtained expressions for the three-dimensional elastostatic Green's functions in anisotropic materials by use of Radon transforms and contour integration in the complex plane for the inversion. Details regarding the evaluation of elastic Green's functions in isotropic and anisotropic solids, along with many applications are included in the review of Bacon *et al.* (1978) and the text of Mura (1987).

In linear piezoelectric solids the electric and elastic response is anisotropic and coupled. Formally, four Green's functions (tensors) exist which describe the elastic displacement and electric potential at an observation point $\mathbf{x}$ due to a unit point force or charge at a source point $\mathbf{x}'$. Physically the four Green's functions represent:

1. the elastic displacement at $\mathbf{x}$ due to a point force at $\mathbf{x}'$
2. the elastic displacement at $\mathbf{x}$ due to a point charge at $\mathbf{x}'$
3. the electric potential at $\mathbf{x}$ due to a point force at $\mathbf{x}'$
4. the electric potential at $\mathbf{x}$ due to a point charge at $\mathbf{x}'$

An interesting phenomena regarding the infinite-body piezoelectric Green's functions that is not completely intuitive is that there is symmetry in the functions 2 and 3 listed above. In particular, the elastic displacement at $\mathbf{x}$ in the $x_m$ direction due to a negative unit point charge at $\mathbf{x}'$ is equal to the electric potential at $\mathbf{x}$ due to a unit point force at $\mathbf{x}'$ in the $x_m$ direction. That this is the case is apparent from many of the below-mentioned works, but it has not been explicitly discussed. It is interesting because it is not even immediately apparent that the units of these two quantities are the same.

Unlike in the case of anisotropic elasticity, relatively little work has been done regarding the study of Green's functions in piezoelectric solids. The most significant work is that of Deeg (1980) who used transform techniques to obtain a contour integral representation of the piezoelectric Green's functions. Deeg's contour integral representation is analogous to that obtained by Synge (1957) in anisotropic elasticity. More recent studies regarding Green's functions in piezoelectric solids have been carried out by Wang (1992), Benveniste

(1992), and Chen (1993) who all used Fourier transform formalism to obtain representations of the Green's functions in the transformed domain, however, neither inverted the transforms to obtain explicit representations for the Green's functions. Chen and Lin (1993) presented a numerical algorithm to compute the derivatives of the piezoelectric Green's functions, the derivatives being based on the anisotropic elasticity formalism of Barnett (1972). Dunn (1994) obtained explicit expressions for the Green's functions for transversely-isotropic piezoelectric solids through use of the Radon transform with the inversion accomplished by contour integration in the complex plane. Finally, Lee and Jiang (1994) and Denda (1994) have obtained expressions for the two-dimensional Green's functions in piezoelectric solids, also through use of Fourier transforms.

In this work we derive explicit, closed-form expressions for the infinite-body Green's functions for a transversely isotropic piezoelectric solid. Transverse isotropy is the symmetry of poled piezoelectric ceramics, perhaps the most technologically-important class of piezoelectric materials. We obtain explicit expressions for the Green's functions by first developing a general solution procedure in terms of piezoelectric potentials. Following along the lines of the Lekhnitskii (1940)-Hu (1953) formalism for transversely isotropic elasticity, we express the general solution for axially-symmetric problems in transversely isotropic piezoelectricity in terms of a single potential, $g$, that satisfies a weighted triharmonic equation. For general problems in transversely isotropic piezoelectricity we express the complete electroelastic solution in terms of two potentials: $g$ and a second potential $\psi$ that satisfies a weighted harmonic equation. A similar potential formulation has recently been proposed by Wang and Zheng (1995) who used it to obtain the solution for a concentrated shear load on the surface of a transversely isotropic piezoelectric half space. By assuming a series solution for the potentials we derive explicit expressions for the piezoelectric Green's functions. When piezoelectric coupling is absent, our solutions reduce to Pan and Chou's (1976) transversely isotropic elasticity Green's functions and the corresponding transversely isotropic electrostatic Green's function.

The transversely isotropic Green's functions studied here are the same as those we considered previously (Dunn, 1994), but the approach used here is quite different, and the results are significantly simpler. They are also easier to implement numerically, such as in a boundary element program, than our previous results because they do not exhibit the deficiency suffered by that solution where each individual term of a component of the Green's function tends to infinity as the $z$-axis is approached. The results presented here are additionally advantageous as compared to our previous results because they can be used to obtain explicit closed-form expressions for the piezoelectric analog of Eshelby's (1957) tensor for ellipsoidal inclusions. We were not able to do this using our previous solution. We present these results in another publication (Dunn and Wienecke, 1996).

## 2. EQUATIONS OF LINEAR PIEZOELECTRICITY

Here, a three-dimensional cartesian coordinate system is adopted where position is denoted by the vector $\mathbf{x}$ or $x_i$. In this paper both conventional indicial notation and traditional cartesian $x, y, z$ notation are utilized. For stationary behavior in the absence of free electric charge or body forces, the field equations of linear piezoelectricity consist of the following:

*Divergence equations*:

$$\sigma_{ij,j} = 0$$
$$D_{i,i} = 0 \tag{1}$$

$\sigma_{ij}$ and $D_i$ are the stress and electric displacement, respectively. These equations are the elastic equilibrium equations and Gauss' law of electrostatics, respectively.

*Constitutive equations*:

$$\sigma_{ij} = C_{ijmn}\varepsilon_{mn} - e_{nij}E_n$$
$$D_i = e_{imn}\varepsilon_{mn} + \kappa_{in}E_n \tag{2}$$

Here, $\varepsilon_{ij}$ is the strain, $E_i$ the electric field, and $C_{ijmn}$, $e_{nij}$ and $\kappa_{in}$ are the elastic moduli (measured in a constant electric field), the piezoelectric coefficients (measured at a constant strain or electric field) and the dielectric constants (measured at a constant strain), respectively. The symmetry conditions satisfied by the electroelastic moduli are given by Nye (1957) and we note that $C_{ijmn}$ and $\kappa_{in}$ are positive definite.

*Gradient equations*:

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i})$$
$$E_i = -\phi_{,i} \tag{3}$$

$u_i$ and $\phi$ are the elastic displacement and electric potential, respectively.

For a transversely isotropic piezoelectric solid, the constitutive eqns (2) simplify considerably and are given in the well-known Voigt two-index notation (reverting to $x, y, z$ notation) as:

$$\begin{aligned}
\sigma_{xx} &= C_{11}\varepsilon_{xx} + (C_{11} - 2C_{66})\varepsilon_{yy} + C_{13}\varepsilon_{zz} - e_{31}E_z \\
\sigma_{yy} &= (C_{11} - 2C_{66})\varepsilon_{xx} + C_{11}\varepsilon_{yy} + C_{13}\varepsilon_{zz} - e_{31}E_z \\
\sigma_{zz} &= C_{13}(\varepsilon_{xx} + \varepsilon_{yy}) + C_{33}\varepsilon_{zz} - e_{33}E_z \\
\sigma_{yz} &= 2C_{44}\varepsilon_{yz} - e_{15}E_y \\
\sigma_{xz} &= 2C_{44}\varepsilon_{xz} - e_{15}E_x \\
\sigma_{xy} &= 2C_{66}\varepsilon_{xy} \\
D_x &= 2e_{15}\varepsilon_{xz} + \kappa_{11}E_x \\
D_y &= 2e_{15}\varepsilon_{yz} + \kappa_{11}E_y \\
D_z &= e_{31}(\varepsilon_{xx} + \varepsilon_{yy}) + e_{33}\varepsilon_{zz} + \kappa_{33}E_z
\end{aligned} \tag{4}$$

Using the gradient equations to eliminate $\varepsilon_{ij}$ and $E_i$ in favor of $u_i$ $(u, v, w)$ and $\phi$, we substitute eqns (4) into eqns (1) to express the equilibrium equations and Gauss' law in terms of elastic displacements and the electric potential:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ -a_{12} & a_{22} & a_{23} & a_{24} \\ -a_{13} & a_{23} & a_{33} & a_{34} \\ -a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix} \begin{bmatrix} \phi \\ u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{5}$$

where the components of the $4 \times 4$ matrix of differential operators are:

$$a_{11} = \kappa_{11}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \kappa_{33}\frac{\partial^2}{\partial z^2}$$

$$a_{12} = -(e_{15} + e_{31})\frac{\partial^2}{\partial x \partial z}$$

$$a_{13} = -(e_{15} + e_{31})\frac{\partial^2}{\partial y \partial z}$$

$$a_{14} = -e_{15}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)-e_{33}\frac{\partial^2}{\partial z^2}$$

$$a_{22} = C_{11}\frac{\partial^2}{\partial x^2}+C_{66}\frac{\partial^2}{\partial y^2}+C_{44}\frac{\partial^2}{\partial z^2}$$

$$a_{23} = (C_{11}-C_{66})\frac{\partial^2}{\partial x\partial y}$$

$$a_{24} = (C_{13}+C_{44})\frac{\partial^2}{\partial x\partial z}$$

$$a_{33} = C_{66}\frac{\partial^2}{\partial x^2}+C_{11}\frac{\partial^2}{\partial y^2}+C_{44}\frac{\partial^2}{\partial z^2}$$

$$a_{34} = (C_{13}+C_{44})\frac{\partial^2}{\partial y\partial z}$$

$$a_{44} = C_{44}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+C_{33}\frac{\partial^2}{\partial z^2} \tag{6}$$

## 3. GENERAL SOLUTION PROCEDURE

We begin by seeking a general solution to eqns (5). The elastic equilibrium equations (the last three of eqns (5)) can be satisfied by expressing $u, v, w$, and $\phi$ in terms of a piezoelectric potential function $g$:

$$u = -\begin{vmatrix} -a_{12} & a_{23} & a_{24} \\ -a_{13} & a_{33} & a_{34} \\ -a_{14} & a_{34} & a_{44} \end{vmatrix} g$$

$$v = \begin{vmatrix} -a_{12} & a_{22} & a_{24} \\ -a_{13} & a_{23} & a_{34} \\ -a_{14} & a_{24} & a_{44} \end{vmatrix} g$$

$$w = -\begin{vmatrix} -a_{12} & a_{22} & a_{23} \\ -a_{13} & a_{23} & a_{33} \\ -a_{14} & a_{24} & a_{34} \end{vmatrix} g$$

$$\phi = \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{23} & a_{33} & a_{34} \\ a_{24} & a_{34} & a_{44} \end{vmatrix} g \tag{7}$$

This potential function representation extends that of Hu (1953) in transversely isotropic elasticity. It can, however, be applied to general linear systems of differential equations: the prescription is obvious from the above and can be immediately verified. If we expand eqns (7) we find that the equations contain common differential operators. After factoring out the common operators, $u, v, w$, and $\phi$ can be expressed as:

$$u = \left[(C_{13}e_{15}-C_{44}e_{31})\frac{\partial^2}{\partial x\partial z}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+[(C_{44}+C_{13})e_{33}-C_{33}(e_{15}+e_{31})]\frac{\partial^4}{\partial x\partial z^3}\right]g$$

$$v=\left[(C_{13}e_{15}-C_{44}e_{31})\frac{\partial^2}{\partial y\partial z}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+[(C_{44}+C_{13})e_{33}-C_{33}(e_{15}+e_{31})]\frac{\partial^4}{\partial y\partial z^3}\right]g$$

$$w=\left[-C_{11}e_{15}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)-C_{44}e_{33}\frac{\partial^4}{\partial z^4}\right.$$
$$\left.+[C_{13}(e_{15}+e_{31})+C_{44}e_{31}-C_{11}e_{33}]\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right]g$$

$$\phi=\left[C_{44}C_{11}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+C_{44}C_{33}\frac{\partial^4}{\partial z^4}\right.$$
$$\left.+(C_{11}C_{33}-2C_{44}C_{13}-C_{13}^2)\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right]g \tag{8}$$

Substituting eqns (8) into Gauss' law (the first of eqns (5)), followed by substantial manipulation, we obtain the following equation that must be satisfied by $g$:

$$\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^3+\frac{a}{d}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2\frac{\partial^2}{\partial z^2}+\frac{b}{d}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{\partial^4}{\partial z^4}+\frac{c}{d}\frac{\partial^6}{\partial z^6}\right]g=0 \tag{9}$$

where:

$$a=C_{11}(\kappa_{11}C_{33}+2e_{15}e_{33})-\kappa_{11}C_{13}(C_{13}+2C_{44})+C_{44}(\kappa_{33}C_{11}+e_{31}^2)-2e_{15}C_{13}(e_{31}+e_{15})$$
$$b=C_{33}[\kappa_{11}C_{44}+\kappa_{33}C_{11}+e_{31}(e_{31}+e_{15})]-C_{13}\kappa_{33}(C_{13}+2C_{44})$$
$$+(e_{31}+e_{15})(C_{33}e_{15}-2C_{13}e_{33})+e_{33}(C_{11}e_{33}-2C_{44}e_{31})$$
$$c=C_{44}(\kappa_{33}C_{33}+e_{33}^2)$$
$$d=C_{11}(\kappa_{11}C_{44}+e_{15}^2) \tag{10}$$

We recognize that eqn (9) can be factored as:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{1}{v_1^2}\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{1}{v_2^2}\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{1}{v_3^2}\frac{\partial^2}{\partial z^2}\right)g=0 \tag{11}$$

where $-1/v_1^2$, $-1/v_2^2$, and $-1/v_3^2$ are the roots of the cubic equation:

$$s^3+\frac{a}{d}s^2+\frac{b}{d}s+\frac{c}{d}=0 \tag{12}$$

The three $v_i$ are easily obtained using the well-known solution for a cubic equation. The roots are either real or one is real and the other two are complex conjugates. The roots are functions only of the material properties of the transversely isotropic solid. With this formalism, the problem solving the three equilibrium equations and Gauss' law has been reduced to solving the $z$-weighted triharmonic eqn (11) for the potential $g$.

At this point, our solution can be viewed as a generalization of that of Lekhnitskii (1940) for axially-symmetric problems in transversely isotropic elastic media. Since the potential $g$ satisfies a sixth-order differential equation, the displacements and electric potential defined by eqns (8) can satisfy only three boundary conditions. The complete solution for $u, v, w$, and $\phi$ must satisfy four boundary conditions, so the above solution is not complete. To solve the general problem we follow Hu (1953) and introduce a second potential $\psi$ so that:

$$u = \left[ (C_{13}e_{15} - C_{44}e_{31})\frac{\partial^2}{\partial x\,\partial z}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + [(C_{44}+C_{13})e_{33} - C_{33}(e_{15}+e_{31})]\frac{\partial^4}{\partial x\,\partial z^3} \right] g - \frac{\partial \psi}{\partial y}$$

$$v = \left[ (C_{13}e_{15} - C_{44}e_{31})\frac{\partial^2}{\partial y\,\partial z}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + [(C_{44}+C_{13})e_{33} - C_{33}(e_{15}+e_{31})]\frac{\partial^4}{\partial y\,\partial z^3} \right] g + \frac{\partial \psi}{\partial x} \tag{13}$$

while $w$ and $\phi$ are still given by eqns (8). Substituting $u$ and $v$ of eqns (13) and $w$ and $\phi$ of eqns (8) into eqns (5) reveals that $g$ must again satisfy eqn (11) and $\psi$ must satisfy the following $z$-weighted harmonic equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{v_0^2}\frac{\partial^2}{\partial z^2}\right)\psi = 0 \tag{14}$$

where $v_0 = \sqrt{C_{66}/C_{44}}$. Thus the general solution to a problem in transversely-isotropic piezoelectricity is reduced to finding the two potentials $g$ and $\psi$ that satisfy the $z$-weighted triharmonic and harmonic eqns (11) and (14), respectively.

## 4. PIEZOELECTRIC GREEN'S FUNCTIONS

The complete set of piezoelectric Green's functions can be obtained from the solution for $u, v, w$, and $\phi$ for three problems:

1. a point force at the origin directed along the $z$-axis
2. a point charge at the origin
3. a point force at the origin directed along the $x$-axis.

Of course, once these solutions are obtained, the more general solution for arbitrary location and direction of the point force and charge are easily obtained. In the following we treat problems 1 and 2 together because they are both axially symmetric. We then solve problem 3.

To obtain the electroelastic Green's functions we must find the potentials $g$ and $\psi$ that give rise to the displacements $(u, v, w)$ and electric potential $(\phi)$ that satisfy the following conditions:

1. $u, v, w$, and $\phi$ vary as $1/r$ for $r \to 0$ where $r^2 = x^2+y^2+z^2$.
2. $u, v, w$, and $\phi$ vanish at infinity.
3. The resultant stresses acting on the surface of an infinitesimal cavity centered at the origin are equivalent to the applied point force.
4. The resultant electric displacement acting on the surface of an infinitesimal region centered at the origin is equivalent to the applied point charge.

That the first condition is the case is evident from the general expressions of the Green's functions obtained using transform formalism. Specifically, the Green's functions, for arbitrary anisotropy, can be expressed as the product of two terms: a position-dependent term that varies as $1/r$ and an orientation-dependent term (Deeg, 1980; Dunn, 1994).

### 4.1. *Point charge and point force in z-direction*

To obtain the electroelastic solution for this axially-symmetric problem, we set $\psi = 0$ and assume a solution for $g$ of the form:

$$g = \sum_{i=1}^{3} A_i f(x, y, z_i) = \sum_{i=1}^{3} A_i [Q_1(x, y, z_i) z_i \ln R_i^* + Q_2(x, y, z_i) R_i + Q_3(x, y, z_i) z_i] \quad (15)$$

where $A_i$ are undetermined coefficients and:

$$R_i = \sqrt{x^2 + y^2 + z_i^2}$$
$$R_i^* = R_i + z_i$$
$$z_i = v_i z \quad (16)$$

In eqn (15), $Q_j(x, y, z_i)$ are quadratic polynomials in $x, y$, and $z_i$. In particular:

$$Q_j(x, y, z_i) = q_{j0} + q_{j1}x + q_{j2}y + q_{j3}z_i + q_{j4}xy + q_{j5}xz_i + q_{j6}yz_i + q_{j7}x^2 + q_{j8}y^2 + q_{j9}z_i^2 \quad (17)$$

The assumed form of the potential $g$ given by eqn (15) gives rise to displacements and an electric potential that satisfy requirements 1 and 2 above.

The $f(x, y, z_i)$ must satisfy the $z$-weighted harmonic:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{v_i^2}\frac{\partial^2}{\partial z^2}\right) f = 0 \quad (18)$$

To determine the explicit form of $f(x, y, z_i)$ we determine the thirty $q_{ij}$ required to make eqn (18) hold. After considerable computation, $f(x, y, z_i)$ can be expressed in the relatively simple form:

$$f(x, y, z_i) = 3[3R_i^2 - 5z_i^2] z_i \ln R_i^* - [4R_i^2 - 15z_i^2] R_i - 6z_i^3 \quad (19)$$

Since the $f(x, y, z_i)$ of eqn (19) satisfy eqn (18), the potential $g$ formed in the manner of eqn (15) identically satisfies eqn (11). With $g$ obtained from eqns (15) and (19), eqns (8) can be used to determine the elastic displacements and electric potential:

$$u = \sum_{i=1}^{3} A_i \lambda_i^{uv} \frac{x}{R_i R_i^*}$$
$$v = \sum_{i=1}^{3} A_i \lambda_i^{uv} \frac{y}{R_i R_i^*}$$
$$w = \sum_{i=1}^{3} A_i \lambda_i^{w} \frac{1}{R_i}$$
$$\phi = \sum_{i=1}^{3} A_i \lambda_i^{\phi} \frac{1}{R_i} \quad (20)$$

where:

$$\lambda_i^{uv} = [(C_{13} + C_{44})e_{33} - C_{33}(e_{31} + e_{15})]v_i^3 + (C_{44}e_{31} - C_{13}e_{15})v_i$$
$$\lambda_i^{w} = -C_{44}e_{33}v_i^4 - [e_{31}(C_{13} + C_{44}) - e_{33}C_{11} + e_{15}C_{13}]v_i^2 - e_{15}C_{11}$$
$$\lambda_i^{\phi} = C_{44}C_{33}v_i^4 + [C_{13}(C_{13} + 2C_{44}) - C_{11}C_{33}]v_i^2 + C_{44}C_{11} \quad (21)$$

The constants in eqns (21) are only functions of the electroelastic moduli of the solid. We also note that $u/x = v/y$.

Requiring that $u$ and $v$ are bounded on the $-z$ axis leads to:

$$\sum_{i=1}^{3} A_i \lambda_i^{uv} = 0 \tag{22}$$

The remaining two equations needed to solve for the three $A_i$ are obtained from requirements 3 and 4 above. The stress and electric displacements are determined from the $u, v, w$, and $\phi$ of eqns (20) through use of the gradient eqns (3) and the constitutive eqns (4). The force and charge balances of requirements 3 and 4 are enforced by integrating the traction and normal component of the electric displacement over the surface of a small spherical cavity centered at the origin, and requiring these to balance the point force in the $z$-direction, $P_z$, and the point charge, $Q$. These balances lead to:

$$\sum_{i=1}^{3} A_i \frac{n_i^a}{v_i^2 - 1} = \frac{P_z}{2\pi} \tag{23}$$

$$\sum_{i=1}^{3} A_i \frac{n_i^e}{v_i^2 - 1} = \frac{Q}{2\pi} \tag{24}$$

where:

$$\begin{aligned} n_i^a &= 2[\lambda_i^{uv}(C_{13} + C_{44} v_i^2) + v_i \lambda_i^w (C_{44} - C_{33}) + v_i \lambda_i^\phi (e_{15} - e_{33})] \\ n_i^e &= 2[-\lambda_i^{uv}(e_{15} v_i^2 + e_{31}) + v_i \lambda_i^w (e_{33} - e_{15}) + v_i \lambda_i^\phi (\kappa_{11} - \kappa_{33})] \end{aligned} \tag{25}$$

We retain $P_z$ and $Q$ but note that the Green's functions are obtained by setting $P_z$ and $Q$ to unity. Equations (22)–(24) are a linear system that can easily be solved for the three $A_i$. We present the results for the two cases of a point force and a point charge separately.

*Point force in the z-direction:*

$$\begin{aligned} A_1 &= (v_1^2 - 1)[n_2^e \lambda_3^{uv}(v_3^2 - 1) - n_3^e \lambda_2^{uv}(v_2^2 - 1)] \frac{P_z}{2\pi\gamma_a} \\ \gamma_a &= (v_1^2 - 1)\lambda_1^{uv}(n_2^a n_3^e - n_3^a n_2^e) + (v_2^2 - 1)\lambda_2^{uv}(n_3^a n_1^e - n_1^a n_3^e) + (v_3^2 - 1)\lambda_3^{uv}(n_1^a n_2^e - n_2^a n_1^e) \end{aligned} \tag{26}$$

*Point charge:*

$$\begin{aligned} A_1 &= \frac{(v_1^2 - 1)(v_2^2 - 1)(v_3^2 - 1)}{v_1(v_1^2 - v_2^2)(v_1^2 - v_3^2)} \frac{(-Q)}{4\pi\gamma_e} \\ \gamma_e &= (\kappa_{11} - \kappa_{33})[C_{11}(C_{44} - C_{33}) + C_{44}(C_{33} + 2C_{13}) + C_{13}^2] + C_{11}(e_{33} - e_{15})^2 \\ &\quad + C_{33}(e_{31} + e_{15})^2 - C_{44}(e_{33} + e_{31})^2 + 2C_{13}[e_{15}(e_{15} + e_{31} - e_{33}) - e_{33}e_{31}] \end{aligned} \tag{27}$$

For both cases, $A_2$ and $A_3$ are obtained from $A_1$ by cyclically permuting the indices 1, 2, and 3.

### 4.2. *Point force in x-direction*

Following a similar procedure as used for the point force in the $z$-direction, we assume solutions for the potentials $g$ and $\psi$ in the forms:

$$g = \sum_{i=1}^{3} D_i \left[ Q_4(x,y,z_i)x \ln R_i^* + Q_5(x,y,z_i)\frac{xz_i}{R_i^*} + Q_6(x,y,z_i)x \right]$$

$$\psi = \frac{-D_0 y}{R_0^*} \tag{28}$$

Again the $Q_j$ are polynomials as defined in eqn (17). We solve for the coefficients of the polynomials $Q_j$ in the same manner as for the axially-symmetric problem. After much manipulation the potential $g$ can then be expressed as:

$$g = \sum_{i=1}^{3} D_i \left[ 3[R_i^2 - 5z_i^2]x \ln R_i^* + \frac{[13R_i^2 - 15z_i^2]xz_i}{R_i^*} + 4xz_i^2 \right] \tag{29}$$

With $g$ and $\psi$ obtained from eqns (28) and (29), eqns (8) and (13) can be used to determine the elastic displacements and electric potential:

$$u = D_0 \left[ \frac{1}{R_0^*} - \frac{y^2}{R_0 R_0^{*2}} \right] - \sum_{i=1}^{3} D_i \lambda_i^{uv} \left[ \frac{1}{R_i^*} - \frac{x^2}{R_i R_i^{*2}} \right]$$

$$v = D_0 \frac{xy}{R_0 R_0^{*2}} + \sum_{i=1}^{3} D_i \lambda_i^{uv} \frac{xy}{R_i R_i^{*2}}$$

$$w = \sum_{i=1}^{3} D_i \lambda_i^{w} \frac{x}{R_i R_i^*}$$

$$\phi = \sum_{i=1}^{3} D_i \lambda_i^{\phi} \frac{x}{R_i R_i^*} \tag{30}$$

The unknown coefficients $D_0, D_1, D_2$, and $D_3$ can be obtained by enforcing the requirement that the solution be bounded on the $-z$-axis, along with the force balance condition. Requiring that $u$ and $v$ be bounded on the $-z$-axis leads to the equation:

$$D_0 v_0 + \sum_{i=1}^{3} D_i v_i \lambda_i^{uv} = 0 \tag{31}$$

Requiring that $w$ and $\phi$ be bounded on the $-z$-axis leads to the two equations:

$$\sum_{i=1}^{3} D_i \lambda_i^{w} = 0$$

$$\sum_{i=1}^{3} D_i \lambda_i^{\phi} = 0 \tag{32}$$

The solution of eqns (30) identically satisfies the charge balance requirement discussed above. The force balance then results in:

$$D_0 v_0 C_{44} + \sum_{i=1}^{3} D_i \frac{n_i^t}{v_i^2 - 1} = \frac{P_x}{2\pi} \tag{33}$$

where:

$$n_i^t = v_i\lambda_i^{uv}(C_{44}-C_{11})+\lambda_i^w(C_{13}v_i^2+C_{44})+\lambda_i^\phi(e_{31}v_i^2+e_{15}) \tag{34}$$

Equations (31)–(33) are a linear system that can easily be solved for the four $D_i$. The solutions can be expressed as:

$$D_0 = \frac{P_x}{4\pi C_{44}v_0}$$

$$D_1 = \frac{(\lambda_2^\phi\lambda_3^w-\lambda_3^\phi\lambda_2^w)}{C_{44}}\frac{P_x}{4\pi\gamma_t}$$

$$\gamma_t = v_1\lambda_1^{uv}(\lambda_3^\phi\lambda_2^w-\lambda_2^\phi\lambda_3^w)+v_2\lambda_2^{uv}(\lambda_1^\phi\lambda_3^w-\lambda_3^\phi\lambda_1^w)+v_3\lambda_3^{uv}(\lambda_2^\phi\lambda_1^w-\lambda_1^\phi\lambda_2^w) \tag{35}$$

$D_2$ and $D_3$ are obtained from $D_1$ by cyclically permuting the indices 1, 2, and 3.

### 4.3. *Discussion*

The explicit, closed-form solutions to the three key problems presented in the previous two sections suffice to determine the complete Green's function tensors for transversely isotropic piezoelectric solids. Once the displacements and electric potential are computed by the equations of the previous section, the strains and electric field can immediately be computed from the gradient eqns (3). The stresses and electric displacement can then easily be computed using the constitutive eqns (4). To validate our solution, we have verified that these stresses and the electric displacement identically satisfy the divergence eqns (1). In addition, we verified that in the absence of piezoelectric coupling, our solution recovers the transversely isotropic elastic Green's functions (Pan and Chou, 1976) and the corresponding electrostatic Green's function for a transversely-isotropic dielectric. Our solution agrees, numerically, with that of Dunn (1994) obtained through use of Radon transforms and contour integration. Due to the complexity of that solution, it is difficult to show analytically that the two solutions agree. In fact, that is part of the motivation for the present solution. Finally, our solution satisfies the Euler relations for $u, v, w$, and $\phi$ given by Chen and Lin (1993).

## 5. CONCLUSION

We derived closed-form expressions for the infinite-body Green's functions for a transversely isotropic piezoelectric medium using a potential formulation where the three displacements and the electric potential are derivable from two potential functions. For axially-symmetric problems, all four quantities are derivable from a single potential function. The simplicity of our closed-form expressions for the piezoelectric Green's functions renders them immediately useful for the analysis of many problems in the mechanics and physics of piezoelectric solids.

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